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Stability of subdivision schemes

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Abstract

The stability of stationary interpolatory subdivision schemes for univariate data is investigated. If the subdivision scheme is linear, its stability follows from the convergence of the scheme, but for nonlinear subdivision schemes one needs stronger conditions and the stability analysis of nonlinear schemes is more involved. Apart from the fact that it is natural to demand that subdivision schemes are stable, it also has an advantage in a theoretical sense: is it shown that the approximation properties of stable schemes can very easily be determined.

Keywords: subdivision, stability, approximation order, computer aided geometric design

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1 Introduction

The method of representing a function, curve or surface by means of a discrete set of data, on which a certain subdivision scheme is applied in order to generate a more refined view on this object is by now a well established technique in computer aided geometric design (CAGD). Subdivision methods are fast and cheap, and they are local. This last property has as main advantage that a local change in the data will only have a local effect on the resulting object, which is of course an attractive feature in designing an object. For example, spline interpolation methods are in general not local as systems of equations have to be solved.

A possible drawback of subdivision schemes is that one wants in practice to represent objects in a smooth way (at least C^1), and analysing the smoothness of the limit function using a subdivision scheme is more difficult than

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e.g., determining the smoothness of a spline. For *linear* subdivision schemes however, the analysis has been highly developed, see *e.g.*, [Dub86], [DGL87], [DD89], [CM89], [CDM91], [DGL91], [Dyn92]. Unfortunately, many results for linear subdivision schemes do not apply to nonlinear schemes, and the proofs are considerably more complicated, see *e.g.*, [KvD98a], [KvD97b], [KvD97a], [KvD98b].

Apart from the smoothness of the result of an approximation method, another important issue is whether the approximative method is also stable. Stability is a well-known concept within numerical analysis:

Definition 1 (Stability of an algorithm) *Given an algorithm A, that depends on certain data x, with result $R_A(x)$. We call A stable if one can bound the change of R_A when the input is perturbed with a small quantity δx, in the following way:*

$$\|R_A(x + \delta x) - R_A(x)\|_\infty \leq C_1 \cdot \|\delta x\|_\infty, \;\; C_1 < \infty.$$

Note that if an algorithm is stable according to this definition, it need not be *numerically stable*: in the latter case C_1 should not be 'too large'.

So the question is legitimate whether a subdivision scheme is stable or not, and this property is of course important in real applications: in designing an object one does not want major changes if one slightly distorts the initial data.

It turns out that for *linear* subdivision schemes the relation between the smoothness of such a scheme and its stability is clear-cut. We will show, that if the scheme converges, *i.e.*, the limit function is C^0, the algorithm is also stable. For *nonlinear* schemes however, it seems not to be possible to prove such a general statement, and whether a nonlinear subdivision scheme is stable or not, requires much deeper analysis.

Of course, in practice one only wants to use stable subdivision schemes, and therefore one possibility would be to restrict ourselves to linear schemes that are smooth (at least C^1). However, such schemes can never be shape preserving. Indeed, smoothness properties of linear subdivision schemes do not depend on the initial data, and therefore linear schemes can never be *e.g.*, convexity preserving, as it is possible to construct convex initial data that do not allow *any* C^1 function that interpolates that data. For example, one can draw data from $f(x) = |x|$ including the point $(0, 0)$.

A subdivision scheme can be viewed upon as an approximation method. Apart from the stability, as well as the smoothness of the results of any approximation method, its approximation properties are also of interest. These approximation properties are usually characterised by the *approximation order* of the method:

Definition 2 (Approximation order) *Consider the equidistant univariate data set* $\{(x_i = ih, f_i) \in \mathbb{R}^2\}_{i=0}^N$, *with* $h \cdot N = 1$. *The data values* f_i *are drawn from a function* $f \in C^p([0,1])$, *such that* $f_i = f(x_i)$, $i = 0, \dots, N$. *The function* u_h *is defined as the solution of the approximation method to the given data. Then, the approximation method has approximation order* p, *if*

$$\|u_h - f\|_{[0,1],\infty} \leq C_2 h^p,$$

for a constant C_2 *that does not depend on* h.

The main result of this article is that the approximation order of a stable subdivision scheme is very easy to determine.

This article is organised as follows. In section 2 we give the subdivision schemes in which we are interested. Next we prove stability for linear convergent subdivision schemes, and we give examples of the proof of stability for the nonlinear subdivision schemes presented in section 2. In section 4 we prove that a simple algorithm to determine the approximation order for stable subdivision schemes, exists. The effort to obtain approximation orders for existing subdivision schemes in the way we present it here is much simpler than earlier estimates, compare [DGL87], [KvD98a], [KvD97a].

2 Subdivision

2.1 Definition of subdivision schemes

Given initial data $(x_i^{(0)}, f_i^{(0)})$, with $x_i^{(0)} = i \cdot h$, the purpose of a subdivision scheme is to generate a sequence of nested data $(x_i^{(k)}, f_i^{(k)})$, $k = 1, 2, \dots$. Hopefully this process converges to a limit function, which we denote by $f^{(\infty)}$. Symbolically we write the subdivision scheme on the functional data f as an operator S, *i.e.*, $f^{(k+1)} = S(f^{(k)})$. Usually, subdivision schemes are considered to be *local*, *i.e.*, they use a finite number of neighbouring points. Moreover, if the scheme satisfies $f^{(k)} = S(f^{(k-1)}) = S^{(k)}(f^{(0)})$, it is called *stationary*, which means that the same subdivision rule is applied at any iteration level k, *i.e.*, the scheme itself does not depend on the data.

An important class of subdivision schemes are *binary* schemes. Binary subdivision schemes are discrete algorithms that (roughly) double the number of data in every iteration, and are defined as follows:

$$
\begin{aligned}
f_{2i}^{(k+1)} &= S_1(\{f_{i+j}^{(k)}\}_j), \\
f_{2i+1}^{(k+1)} &= S_2(\{f_{i+j}^{(k)}\}_j).
\end{aligned}
$$

A special class of schemes is obtained by considering *interpolatory* subdivision schemes, which have the property that all data at all subdivision levels remain in the data, *i.e.*, all data are located on the limit function:

$$
\begin{aligned}
f_{2i}^{(k+1)} &= f_i^{(k)}, \\
f_{2i+1}^{(k+1)} &= S_2(\{f_{i+j}^{(k)}\}_j).
\end{aligned} \tag{1}
$$

In this article we assume that the x-data are equidistant. Moreover the x-data are subdivided in the simplest possible way:

$$
\begin{aligned}
x_{2i}^{(k+1)} &= x_i^{(k)}, \\
x_{2i+1}^{(k+1)} &= \frac{1}{2}\left(x_i^{(k)} + x_{i+1}^{(k)}\right).
\end{aligned}
$$

Without loss of generality we start with $x_i^{(0)} = i \cdot h$, and then it directly follows that $x_i^{(k)} = 2^{-k} \cdot i \cdot h$.

In this article we investigate subdivision schemes in the class (1), *i.e.*, the subdivision schemes are local, binary, interpolatory and stationary.

2.2 Interpolatory subdivision schemes

A specific class of subdivision schemes is offered by *linear* subdivision schemes, in which case the function S_2 in (1) is linear in its arguments. As such schemes cannot be shape preserving, we also investigate two nonlinear schemes: the first is convexity preserving, whereas the last preserves monotonicity.

Linear subdivision

One of the simplest interpolatory subdivision scheme is the two-point scheme:

$$
\begin{aligned}
f_{2i}^{(k+1)} &= f_i^{(k)}, \\
f_{2i+1}^{(k+1)} &= \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right).
\end{aligned}
$$

Although this scheme does have shape preserving properties, *e.g.*, it preserves positivity, monotonicity and convexity, the scheme is too trivial for our purposes: it generates the piecewise linear interpolant to the initial data, which is a limit function that is only C^0, and this is usually not smooth enough for practical applications.

In [Dub86], a linear subdivision scheme based on local equidistant cubic interpolation is proposed. This scheme is extended in [DGL87] by including a tension parameter w for shape design. This leads to the well-known linear four-point scheme:

$$\begin{aligned} f_{2i}^{(k+1)} &= f_i^{(k)}, \\ f_{2i+1}^{(k+1)} &= -w f_{i-1}^{(k)} + \left(\frac{1}{2} + w\right) f_i^{(k)} + \left(\frac{1}{2} + w\right) f_{i+1}^{(k)} - w f_{i+2}^{(k)}, \end{aligned} \tag{2}$$

The special case $w = 1/16$, in which case the scheme reproduces cubic polynomials, yields the scheme in [Dub86]. It is proved in [DGL87] that subdivision scheme (2) generates a continuous function if the tension parameter w satisfies $|w| < 1/4$. The scheme converges to C^1 limit functions provided the tension parameter is restricted to the range $0 < w < 1/8$. In more recent articles, convergence and smoothness is proved for a wider range of the tension parameter, however.

It is known that the approximation order of the linear four-point scheme (2) is two, if $|w| < 1/4$. For $w = 1/16$, the scheme has approximation order four, see [DGL87].

Convexity preserving subdivision

A simple C^1 method that preserves convexity can be found in [KvD97b], [KvD98a]:

$$\begin{aligned} f_{2i}^{(k+1)} &= f_i^{(k)}, \\ f_{2i+1}^{(k+1)} &= \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) - F(d_i^{(k)}, d_{i+1}^{(k)}), \end{aligned} \tag{3}$$

where $d_i^{(k)}$ are second differences:

$$d_i^{(k)} = f_{i+1}^{(k)} - 2f_i^{(k)} + f_{i-1}^{(k)}.$$

Convexity is preserved if and only if the function F satisfies

$$0 \leq F(a,b) \leq \frac{1}{4} \min\{a,b\}, \quad \forall a,b \geq 0. \tag{4}$$

In [KvD98a] it was proved that the limit function is C^1 (if the data admit that) if

$$F(a,b) = \frac{1}{4} \frac{1}{\frac{1}{a} + \frac{1}{b}}. \tag{5}$$

The proof essentially uses the also in [KvD98a] proven fact that ratios of second differences remain bounded, independent of the iteration level: there exists a $D < \infty$ such that

$$\max_i \max\{d_i^{(k)}/d_{i+1}^{(k)}, d_{i+1}^{(k)}/d_i^{(k)}\} \leq D, \ \forall k. \tag{6}$$

This condition can be viewed upon as a necessary condition for C^1, and this condition turns out to be vital in the proof of stability of this scheme (see section 3).

In section 4 we discuss the approximation properties of all schemes in this section. However, for convexity preserving approximation methods one can show that they have an approximation order of at least 2:

Theorem 3 *Interpolation methods that preserve convexity are second order accurate.*

PROOF. (Sketch). It is fairly easy to prove that any approximation method that preserves convexity must also yield a continuous result. Then by constructing lower and upper envelopes in which the approximation must lie due to the convexity preservation and interpolation, one easily establishes quadratic precision.

Monotonicity preserving subdivision

A class of schemes that preserve monotonicity can be found in [KvD97a]:

$$\begin{aligned} f_{2i}^{(k+1)} &= f_i^{(k)}, \\ f_{2i+1}^{(k+1)} &= \frac{1}{2}\left(f_i^{(k)} + f_{i+1}^{(k)}\right) + \frac{1}{2}\left(f_{i+1}^{(k)} - f_i^{(k)}\right) G(r_i^{(k)}, R_{i+1}^{(k)}), \end{aligned} \tag{7}$$

where

$$r_i^{(k)} = \frac{f_i^{(k)} - f_{i-1}^{(k)}}{f_{i+1}^{(k)} - f_i^{(k)}} \quad \text{and} \quad R_i^{(k)} = \frac{1}{r_i^{(k)}}.$$

Subdivision scheme (7) preserves monotonicity if and only if the subdivision function G satisfies

$$|G(r,R)| \leq 1, \quad \forall r, R \geq 0.$$

A possible class of functions G for which the resulting limit function is C^1, is given by [KvD97a]:

$$G(r,R) = \frac{r - R}{\ell_1 + (1+\ell_2)(r+R) + \ell_3 rR}, \quad (\ell_1, \ell_2, \ell_3) \in \Omega, \tag{8}$$

where Ω is defined by

$$\Omega = \{(\ell_1, \ell_2, \ell_3) \mid \ \ell_1, \ell_2, \ell_3 \geq 0, \quad \ell_1 + 2\ell_2 + \ell_3 = 6\}. \tag{9}$$

Subdivision scheme (7-9) has the property that the ratios of adjacent first order differences obey [KvD97a]:

$$\exists \rho < 1: \quad \max_i \left| \max \left\{ r_i^{(k)}, \frac{1}{r_i^{(k)}} \right\} - 1 \right| \leq C_3 \rho^k, \quad C_3 < \infty. \tag{10}$$

In the proof of C^1 smoothness this property is a necessary condition, and also here it turns out that the proof of stability requires such a necessary C^1 condition, as we show in the next section.

3 Stability analysis of interpolatory subdivision schemes

To be able to formulate explicit statements on the approximation order of subdivision schemes, we use the notion of stability for subdivision schemes (see definition 1):

Definition 4 *A subdivision scheme is said to be* stable, *if for perturbed data* $\widetilde{f}_i^{(0)}$ *to* $f_i^{(0)}$, *there exists a* $B < \infty$, *such that*

$$|\widetilde{f}_i^{(0)} - f_i^{(0)}| \leq \delta, \ \forall i \quad \Longrightarrow \quad \|\widetilde{f}^{(k)} - f^{(k)}\|_\infty \leq B_k \delta \leq B\delta, \quad \forall \ k.$$

In this section we subsequently treat the stability of linear, convexity preserving and monotonicity preserving subdivision.

Linear subdivision

For linear schemes a weak condition for convergence to a continuous limit functions, is sufficient to prove the stability of the scheme.

Theorem 5 *Consider a linear subdivision scheme that satisfies the sufficient condition for convergence to a continuous limit function:*

$$\exists n \in \mathbb{N}: \quad \|f^{(k+n)} - f^{(k)}\|_\infty \leq C_1 \lambda^k \|f^{(0)}\|_\infty, \quad \lambda < 1, \quad C_1 < \infty. \tag{11}$$

Then, this scheme is stable.

PROOF. If $\widetilde{f}_i^{(0)}$ are the perturbed data, define the data $\widehat{f}_i^{(0)} = \widetilde{f}_i^{(0)} - f_i^{(0)}$. Then, by assumption, it holds that

$$\|\widehat{f}^{(0)}\|_\infty \leq \delta.$$

The sufficient C^0-condition (11) can be applied to the data $f_i^{(0)}$ and $\widetilde{f}_i^{(0)}$, but also to the data $\widehat{f}_i^{(0)}$, *i.e.*,

$$\|\widehat{f}^{(k+n)} - \widehat{f}^{(k)}\|_\infty \leq C_2 \lambda^k \|\widehat{f}^{(0)}\|_\infty \leq C_2 \lambda^k \delta, \quad \lambda < 1,$$

which finally yields

$$\begin{aligned} &\|\widehat{f}^{(\infty)} - \widehat{f}^{(0)}\|_\infty \leq \tfrac{C_2}{1-\sqrt[n]{\lambda}} \delta \quad \Longrightarrow \\ &\|\widehat{f}^{(\infty)}\|_\infty = \|f^{(\infty)} - \widetilde{f}^{(\infty)}\|_\infty \leq \left(1 + \tfrac{C_2}{1-\sqrt[n]{\lambda}}\right) \delta, \end{aligned}$$

which proves stability.

In [DGL87], it has been shown that there exists a $\lambda < 1$ and a $C_3 < \infty$, such that for $|w| < 1/4$ the linear four-point scheme (2) satisfies:

$$\|f^{(k+1)} - f^{(k)}\|_\infty \leq C_3 \lambda \|f^{(0)}\|_\infty.$$

Application of theorem 5 now leads to the following result:

Corollary 6 *The linear four-point scheme (2) is stable if $|w| < 1/4$.*

We now examine stability properties of convexity preserving subdivision schemes. In order to be able to prove stability it is not sufficient that the subdivision scheme yields continuous limit functions: in fact the extra requirement is a necessary (but not sufficient) condition for convergence to a C^1 limit function.

Theorem 7 *Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be C^1 and homogeneous of degree 1 in its arguments. Then interpolatory subdivision schemes in the class (3) which satisfy condition (4) and (6), are stable.*

PROOF. Let the initial data satisfy

$$\max_i |\tilde{f}_i^{(0)} - f_i^{(0)}| \leq \delta \quad \Longrightarrow \quad \max_i |\tilde{d}_i^{(0)} - d_i^{(0)}| \leq 4\delta.$$

We have to prove that

$$\max_i |\tilde{f}_i^{(k)} - f_i^{(k)}| \leq B_k \delta, \; B_k < \infty \quad \text{and} \quad \lim_{k\to\infty} B_k < \infty.$$

We give a proof by induction, and therefore we assume that

$$\max_i |\tilde{f}_i^{(k)} - f_i^{(k)}| \leq B_k \delta.$$

Consider the difference $\|\tilde{f}^{(k+1)} - f^{(k+1)}\|_\infty$, which satisfies the estimate:

$$\begin{aligned} \|\tilde{f}^{(k+1)} - f^{(k+1)}\|_\infty &= \|\tilde{f}^{(k+1)} - \tilde{f}^{(k)} + \tilde{f}^{(k)} - f^{(k)} + f^{(k)} - f^{(k+1)}\|_\infty \\ &\leq \|\tilde{f}^{(k)} - f^{(k)}\|_\infty + \|\tilde{f}^{(k+1)} - \tilde{f}^{(k)} + f^{(k)} - f^{(k+1)}\|_\infty, \end{aligned}$$

and substitution of the definitions yields

$$\|\tilde{f}^{(k+1)} - f^{(k+1)}\|_\infty \leq \|\tilde{f}^{(k)} - f^{(k)}\|_\infty + \max_i |F(d_i^{(k)}, d_{i+1}^{(k)}) - F(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)})|.$$

A Taylor series around $(d_i^{(k)}, d_{i+1}^{(k)})$ yields

$$\begin{aligned} F(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)}) &= F(d_i^{(k)}, d_{i+1}^{(k)}) + F_1\Big(\tau_1 d_i^{(k)} + \tau_2 \tilde{d}_i^{(k)}, \tau_1 d_{i+1}^{(k)} + \tau_2 \tilde{d}_{i+1}^{(k)}\Big) \\ &\cdot |\tilde{d}_i^{(k)} - d_i^{(k)}| + F_2\Big(\tau_1 d_i^{(k)} + \tau_2 \tilde{d}_i^{(k)}, \tau_1 d_{i+1}^{(k)} + \tau_2 \tilde{d}_{i+1}^{(k)}\Big) \cdot |\tilde{d}_{i+1}^{(k)} - d_{i+1}^{(k)}| \end{aligned}$$

for some $0 \leq \tau_1 = 1 - \tau_2 \leq 1$, where F_j denotes the partial derivative of F with respect to its j-th argument.

Convexity preservation demands the inequality (4), but as ratios of second differences are assumed to be bounded with a certain D (possibly depending on the initial data – see (6)) we can easily show that there must exist a $\mu < 1$ such that

$$F(a,b) \le \frac{\mu}{4}\min\{a,b\}.$$

By applying the identity of Euler for homogeneous functions of degree 1:

$$F(a,b) = F_1(a,b)a + F_2(a,b)b,$$

we can get the following estimate. First assume $a \le b$. Then

$$(F_1(a,b)+F_2(a,b))a \le F_1(a,b)a + F_2(a,b)b = F(a,b) \le \frac{\mu}{4}\min\{a,b\} \le \frac{\mu}{4}a,$$

which proves $0 \le F_1(a,b) + F_2(a,b) \le \mu/4$. The case $a \ge b$ can be treated similarly.

This yields

$$\begin{aligned}
&\left|F(d_i^{(k)}, d_{i+1}^{(k)}) - F(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)})\right| \\
&\quad \le \max_i \left|\tilde{d}_i^{(k)} - d_i^{(k)}\right| \cdot \max_{0\le\tau_1\le 1}\Big(F_1\left(\tau_1 d_i^{(k)} + \tau_2\tilde{d}_i^{(k)}, \tau_1 d_{i+1}^{(k)} + \tau_2\tilde{d}_{i+1}^{(k)}\right) \\
&\qquad + F_2\left(\tau_1 d_i^{(k)} + \tau_2\tilde{d}_i^{(k)}, \tau_1 d_{i+1}^{(k)} + \tau_2\tilde{d}_{i+1}^{(k)}\right)\Big) \le \frac{\mu}{4}\max_i\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right|.
\end{aligned}$$

We continue with

$$\begin{aligned}
\left|\tilde{d}_{2i+1}^{(k+1)} - d_{2i+1}^{(k+1)}\right| &= 2\left|F(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)}) - F(d_i^{(k)}, d_{i+1}^{(k)})\right| \le \frac{\mu}{2}\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right| \quad \text{and} \\
\left|\tilde{d}_{2i}^{(k+1)} - d_{2i}^{(k+1)}\right| &\le \frac{1}{2}\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right| + \left|F(\tilde{d}_{i-1}^{(k)}, \tilde{d}_i^{(k)}) - F(d_{i-1}^{(k)}, d_i^{(k)})\right| \\
&\quad + \left|F(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)}) - F(d_i^{(k)}, d_{i+1}^{(k)})\right| \\
&\le \left(\frac{1}{2} + 2\cdot\frac{\mu}{4}\right)\max_i\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right| =: \nu\max_i\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right|,
\end{aligned}$$

with $\nu < 1$. This yields

$$\max_i\left|\tilde{d}_i^{(k)} - d_i^{(k)}\right| \le 4\delta\nu^k,$$

and the conclusion is

$$\|\tilde{f}^{(k+1)} - f^{(k+1)}\|_\infty \le \|\tilde{f}^{(k)} - f^{(k)}\|_\infty + A\nu^k.$$

Therefore $B_{k+1} \leq B_k + A\nu^k$, with $\nu < 1$, which yields that B_k is a bounded sequence.

Remark 8 *Note that the proof essentially uses the locality of* four *points. However, if one considers less local methods which are even C^2 [KvD98b], one can easily prove that such subdivision schemes are also stable: the boundedness of ratios of second order differences remains the essential ingredient.*

Monotonicity preserving subdivision

Next we treat the stability of the monotonicity preserving subdivision schemes (7).

Theorem 9 *Let $G : \mathbb{R}^2 \to \mathbb{R}$ be C^1 and homogeneous of degree 0 in its arguments. Then interpolatory subdivision schemes in the class (7) which preserve strict monotonicity and which satisfy (10), are stable.*

PROOF. (Sketch). The proof follows the same lines as the proof of theorem 7, but as the proof is rather technical, we only give a sketch. For the sake of simplicity, we write $s_i^{(k)} = f_{i+1}^{(k)} - f_i^{(k)}$ and

$$F(s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}) = \frac{1}{2} s_i^{(k)} G(r_i^{(k)}, R_{i+1}^{(k)}).$$

The essential ingredients are that the sum over the *three* partial derivatives of F can be bounded by $\pm\rho/2$ for sufficiently large k. Secondly, Euler's identity gives

$$\left| F(\tilde{s}_{i-1}^{(k)}, \tilde{s}_i^{(k)}, \tilde{s}_{i+1}^{(k)}) - F(s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}) \right| \leq |F_1 + F_2 + F_3| \max_i \left| \tilde{s}_i^{(k)} - s_i^{(k)} \right|,$$

where F_j denotes the partial derivative of F with respect to its j-th argument.

Finally, one can prove that

$$\begin{aligned} \left| \tilde{s}_{2i+j}^{(k+1)} - s_{2i+j}^{(k+1)} \right| &\leq \frac{1}{2} \left| \tilde{s}_i^{(k)} - s_i^{(k)} \right| \pm \left| F(\tilde{s}_{i-1}^{(k)}, \tilde{s}_i^{(k)}, \tilde{s}_{i+1}^{(k)}) + F(s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}) \right| \\ &\leq \left(\frac{1}{2} + \frac{\rho}{2} \right) \max_i \left| \tilde{s}_i^{(k)} - s_i^{(k)} \right|, \end{aligned}$$

which shows stability, as $\rho < 1$.

In this section, it was proved that linear subdivision schemes are stable, as soon as they also generate continuous limit functions. For the shape preserving schemes discussed, we need a little more: both schemes discussed here, need to satisfy a necessary condition for C^1 continuity. In the next section we discuss the relation of the stability with approximation orders of subdivision.

4 Approximation order

The next theorems provide conditions for the approximation order of a subdivision scheme. The theorems apply to linear and nonlinear subdivision schemes.

First, the treatment of the boundaries is examined. Note that, using Taylor series of f in $x_0^{(0)}$ and $x_N^{(0)}$, any function $f \in C^p(I)$ can be extended to a function $\tilde{f} \in C^p(\tilde{I})$, where $\tilde{I} = [-\ell \cdot h, 1+\ell \cdot h])$, such that $\tilde{f}(x) = f(x), \forall x \in I$. Moreover, if f is convex/monotone $\tilde{f}$ is also convex/monotone, provided $\ell \cdot h$ is sufficiently small. The quantity ℓ is related to the locality of the subdivision at hand: for four point schemes, $\ell = 2$. The boundary data points $f_{-\ell}^{(0)}, \ldots, f_{-1}^{(0)}$, $f_{N+1}^{(0)}, \ldots, f_{N+\ell}^{(0)}$, which are necessary to determine the limit function $f^{(\infty)}$ in I, are now drawn from this extended function $\tilde{f}$.

Theorem 10 (Sufficiency) *Let a subdivision scheme be stable and local, and let it reproduce polynomials of degree $p-1$, with $p \geq 1$. Then, the subdivision scheme has approximation order p.*

PROOF. Without loss of generality, consider the interval $I_i = [x_i^{(0)}, x_{i+1}^{(0)}]$. It is necessary and sufficient for approximation order p that

$$\|f_h^{(\infty)} - f\|_{I_i,\infty} \leq C_1 h^p, \quad C_1 < \infty.$$

This can be achieved by defining $\widetilde{f}$ as the $(p-1)$-th degree Taylor polynomial of f at $x = x_{2i+1}^{(1)}$, which obviously satisfies

$$\|f - \widetilde{f}\|_{I_i,\infty} \leq C_2 h^p, \ C_2 < \infty.$$

The subdivision scheme is applied to the (perturbed) data $\widetilde{f}_j^{(0)}$ drawn from $\widetilde{f}$ at the parameters $x_j^{(0)}$, $j = i-\ell, \ldots, i+1+\ell$. As a direct result of the stability, see definition 4, the limit function $\widetilde{f}_h^{(\infty)}$ then satisfies

$$\|f_h^{(\infty)} - \widetilde{f}_h^{(\infty)}\|_{I_i,\infty} \leq C_3 h^p, \quad C_3 < \infty,$$

and since the subdivision scheme reproduces polynomials of degree p, it also holds that

$$\|\widetilde{f}_h^{(\infty)} - \widetilde{f}\|_{I_i,\infty} = 0.$$

This yields

$$\begin{aligned}
\|f_h^{(\infty)} - f\|_{I_i,\infty} &= \|f_h^{(\infty)} - \widetilde{f}_h^{(\infty)} + \widetilde{f}_h^{(\infty)} - \widetilde{f} + \widetilde{f} - f\|_{I_i,\infty} \\
&\leq \|f_h^{(\infty)} - \widetilde{f}_h^{(\infty)}\|_{I_i,\infty} + \|\widetilde{f}_h^{(\infty)} - \widetilde{f}\|_{I_i,\infty} + \|\widetilde{f} - f\|_{I_i,\infty} \\
&\leq C_3 h^p + 0 + C_2 h^p = C_1 h^p,
\end{aligned}$$

which is valid for all i.

A powerful theorem to simple determine approximation orders for stable subdivision schemes is given by the following:

Theorem 11 (Sufficiency and necessity) *Let a subdivision scheme be stable, and let the approximation order after one iteration be equal to* p, i.e.,

$$\|f^{(1)} - f\|_\infty = \max_i |f_{2i+1}^{(1)} - f((i+1/2)h)| \leq C_4 h^p, \ C_4 < \infty.$$

Then this scheme has approximation order p, i.e.,

$$\|f^{(\infty)} - f\|_\infty \leq C_5 h^p, \ C_5 < \infty.$$

PROOF. In order to facilitate the proof, we write $\bar{f}^{(k)}$ as the dataset that coincides with f at the k-th level of iteration, *i.e.*, $\bar{f}_i^{(k)} = f(x_i^{(k)})$, $\forall i$.

We prove that

$$\lim_{k\to\infty} \|f^{(k)} - \bar{f}^{(k)}\|_\infty \leq C_5 h^p \quad , C_5 < \infty.$$

The following estimate is easily obtained:

$$\|f^{(k)} - \bar{f}^{(k)}\|_\infty \leq \|S^{(1)}(f^{(k-1)}) - S^{(1)}(\bar{f}^{(k-1)})\|_\infty + \|S^{(1)}(\bar{f}^{(k-1)}) - \bar{f}^{(k)}\|_\infty.$$

The first term on the right-hand-side can be estimated as

$$\begin{aligned}
\|S^{(1)}(f^{(k-1)}) - S^{(1)}(\bar{f}^{(k-1)})\|_\infty &= \|S^{(2)}(f^{(k-2)}) - S^{(1)}(\bar{f}^{(k-1)})\|_\infty \\
&\leq \|S^{(2)}(f^{(k-2)}) - S^{(2)}(\bar{f}^{(k-2)})\|_\infty + \|S^{(2)}(\bar{f}^{(k-2)}) - S^{(1)}(\bar{f}^{(k-1)})\|_\infty,
\end{aligned}$$

and continuing this process, we arrive at:

$$\begin{aligned}\|f^{(k)} - \bar{f}^{(k)}\|_\infty \leq & \|S^{(k)}(f^{(0)}) - S^{(k)}(\bar{f}^{(0)})\|_\infty \\ & + \sum_{m=1}^{k} \|S^{(m)}(\bar{f}^{(k-m)}) - S^{(m-1)}(\bar{f}^{(k-m+1)})\|_\infty .\end{aligned}$$

The first term being identically zero, the second part is further estimated using the stability of the scheme (see for the symbols B, B_m definition 4)

$$\begin{aligned}\|f^{(k)} - \bar{f}^{(k)}\|_\infty \leq & \sum_{m=1}^{k} B_{m-1} \cdot \|S^{(1)}(\bar{f}^{(k-m)}) - \bar{f}^{(k-m+1)}\|_\infty \\ \leq & \sum_{m=1}^{k} B_{m-1} \cdot C_4 \cdot \Big(\frac{h}{2^{k-m}}\Big)^p \leq B \cdot C_4 h^p \cdot \sum_{m=1}^{k} 2^{(m-k)p} \leq C_5 h^p ,\end{aligned}$$

which completes the proof.

Next we apply both theorems for all subdivision schemes which we treated earlier on.

Linear subdivision

Theorem 10 shows that the scheme proposed in [DGL87] is at least quadratic in precision:

Corollary 12 *The linear four-point scheme (2) with* $|w| < 1/4$ *has at least approximation order two.*

As the linear four-point scheme with $w = 1/16$ is stable, the scheme has at least approximation order four, due to theorems 10 and 5: A simple Taylor expansion shows, using theorem 11, that the following corollary holds:

Corollary 13 *The linear four-point scheme (2) with* $w = 1/16$ *has approximation order four.*

Compare the result from corollary 13 with the original proof in [DGL87].

Convexity preserving subdivision

As any convexity preserving scheme must necessarily reproduce linear functions, the stability of the subdivision scheme (3), (5) is sufficient for second order accuracy of convexity preserving interpolatory subdivision schemes. So the stability proof gives an alternative proof of theorem 3.

In the previous section we examined the stability properties of convexity preserving subdivision schemes. It turned out that under weak technical conditions such schemes are stable. Henceforth, due to theorem 11 the approximation properties of the convexity preserving subdivision scheme (3), (5) can easily be obtained. A simple calculation shows that the scheme reproduces quadratic polynomials, which suggests third order accuracy. However, using Taylor's theorem, one easily finds that the scheme has approximation order four, using theorem 11:

Corollary 14 *The approximation order of subdivision scheme (3), (5) equals four.*

Remark 15 *Observe that this result is only valid for strongly convex data, i.e., data drawn from a function f with $f''(x) > 0$. Numerical experiments show that if the function f is C^4 and convex but not strictly convex, the approximation order is also equal to 4.*

The approximation order can also be obtained by comparing the convex scheme (3), (5) with the linear four-point scheme [DGL87] with $w = 1/16$, see (2), but the analysis is technical, see [KvD98a].

Monotonicity preserving subdivision

Although a simple calculation shows that the scheme only reproduces linear functions (and quadratics in the case $\ell_3 = 0$), a simple Taylor expansion of G shows that the scheme has approximation order four.

Corollary 16 *The approximation order of subdivision scheme (7-9) equals four.*

An alternative but much more involved proof can be found in [KvD97a].

Remark 17 *Observe that this analysis is only valid for strictly monotone data, i.e., data drawn from a function f with $f'(x) > 0$, $\forall x \in I$. Numerical experiments show that if $f'(x) = 0$ for some $x \in I$, the approximation order decreases to 3.*

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